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Decimal System in India

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The decimal notation is an outstanding innovation both in its sheer brilliance of abstract thought and as a practical invention. In the words of the French mathematician Pierre-Simon Laplace (1814 CE):

> It is India that gave us the ingenious method of expressing all numbers by means of ten symbols, each symbol receiving a value of position as well as an absolute value; a profound and important idea which appears so simple to us now that we ignore its true merit. But its very simplicity, the great ease which it has lent to all computations, puts our arithmetic in the first rank of useful inventions; and we shall appreciate the grandeur of this achievement the more when we remember that it escaped the genius of Archimedes and Apollonius, two of the greatest men produced by antiquity. (Quoted in (Dantzig, 1930), p. 19)

Due to the simplicity of the decimal notation, children all over the world can now learn basic arithmetic at an early age. This has been a major factor in the proletarisation of considerable scientific and technical knowledge, earlier restricted only to a gifted few.

For an understanding of the history of this great innovation, we need to make a distinction between the two main familiar forms of the decimal system: the (written) decimal place value notation (decimal notation, in short) and the corresponding (oral) decimal nomenclature.

Forms of Decimal System: Notation and Nomenclature

Our standard decimal notation comprises ten symbols: nine figures called "digits" (or "numerals") for the first nine numbers (1,2,3,4,5,6,7,8,9) and an additional symbol "0" as a placeholder to denote the absence of any of the nine digits. Sometimes, 0 is also regarded as a numeral or a digit. Every number, no matter how large, can be expressed by means of these ten symbols using the "place value" (or "positional value") principle by which a digit d in the rth position (place) from the right is imparted the place value $d \times 10^{r-1}$. For instance, in the number denoted by 2107 in decimal notation, the digit 2 acquires the value "two *thousand* (2×10^3)", while the digit 1 acquires the value "one *hundred*". The Sanskrit word for "digit" is *aṅka* (literally, "mark"), and the term for "place" is *sthāna*.

In our standard verbal decimal nomenclature, each number is expressed through nine word-numerals (one, two,. . ., nine in English) corresponding to the nine digits, and words for the "powers of ten" (like "hundred", "thousand", etc.). Here, the number names for the powers 10^n play the role of the place value principle. The concept of zero as placeholder is not required for the verbal expression of a number. For convenience, some additional derived words may be adopted (like "eleven" for "one and ten", "nineteen" for "nine and ten", "twenty" for "two ten", etc.).

Both the above forms originated and developed in India; the decimal notation evolved by the early centuries of the Common Era while the verbal decimal nomenclature was already in vogue when the *Ṛgveda*, the most ancient extant world literature, was compiled. In addition, there was another system in India of expressing numbers (both orally and in writing) using place value in place of number names for powers of ten but using words in place of symbols for the numerals. The words were usually arranged in

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ascending powers of ten; like 2107 being expressed as "seven-zero-one-two". This system suited the tradition of oral transmission.

The word "decimal" (pertaining to ten) is derived from the Latin *decimalis* (adjective for "ten") and *decem* (ten), resembling the Sanskrit *daśama* (tenth) and *daśan* (ten).

The decimal system is based on the mathematical principle that any natural number can be envisaged as a polynomial-like sum $10^n a_n + \ldots + 100a_2 + 10a_1 + a_0$, where $a_0, a_1, \ldots, a_n$ are numbers between 0 and 9. Numbers can also be represented in bases other than ten; for instance, in the binary representation used in computers, numbers are perceived in the form $2^n a_n + \ldots + 2a_1 + a_0$, where each a_i is either 0 or 1. Such representations involve recursive applications of the well-known "division algorithm" that pervades the later Greek, Indian and modern mathematics. The mathematical sophistication of the decimal system can be glimpsed from the fact that its discovery required a realisation of the above principles.

Antiquity of Decimal Nomenclature

The decimal system in India, at least in its verbal form, is an inheritance from an age of mystic intuition in remote antiquity. Numbers are invariably described in decimal nomenclature in ancient Sanskrit treatises right from the Vedic *Saṁhitā* and the *Brāhmaṇa*. These esoteric treatises contain the current Sanskrit terms for the nine primary numbers: *eka* (1), *dvi* (2), *tri* (3), *catur* (4), *pañca* (5), *ṣaṭ* (6), *sapta* (7), *aṣṭa* (8) and *nava* (9); the derived decuple terms for the first nine multiples of ten: *daśa* (10), *viṁśati* (20), *triṁśat* (30), *catvāriṁśat* (40), *pañcāśat* (50), *ṣaṣṭi* (60), *saptati* (70), *aśīti* (80) and *navati* (90); and terms for powers of ten.

Decimal Enumeration in the Ṛgveda

The principles of decimal enumeration had already been mastered by the time of the *Ṛgveda*, the oldest layer of Vedic literature. The Sanskrit terms for the nine numerals occur several times in the *Ṛgveda*. Among powers of ten, the *Ṛgveda* frequently uses *daśa* (ten), *śata* (hundred) and *sahasra* (thousand); *ayuta* (ten thousand) too is mentioned in a few hymns. For compound numbers, the names for nine numerals and powers of ten are combined as in our verbal decimal terminology; e.g., "seven hundred and twenty" is expressed as *sapta śatāni viṁśatiḥ* in *Ṛgveda* (I.164.11). In a detailed study of decimal enumeration in the *Ṛgveda*, B. Bavare and P. P. Divakaran have shown (2013) that this combination is actually an application of the more general grammatical rules of nominal composition followed in Sanskrit from the Vedic time (at least) and formulated later by Pāṇini. Thus, the verbal decimal terminology – the precursor to, and a certain manifestation of, the abstract place value principle – is an offshoot of the grammatical principles of Vedic Sanskrit.

Mystic Significance and Etymology of Powers of Ten

The choice of base ten and the coining of number names for certain powers of ten might have arisen from the special mystic significance attached to these numbers in an ancient era (Dutta, 2008). In the perception of the Vedic seers, the number hundred represents a general fullness; ten layers of hundred symbolises an entire plenitude, an absolute completeness, and a word for plentiful – *sahasra*, derived from the root *sahas* (mighty, powerful, plentiful, forceful, etc.), was adopted for it. Ten thousand was considered special as the mystics saw in the illumined mind ten subtle powers each having an entire plenitude (Sri Aurobindo, 1972, pp. 301–302, 416).

Regarding the etymology of *daśa* (ten), the ancient lexicographer Yāska (sixth century BCE or earlier) says: *daśa dastā dṛṣṭārthā vā* – *daśa* is so called as it closes off a sequence of numbers (the nine numerals) and as its effect can be seen (in forming the next sequence). It may be recalled that similar-sounding Vedic

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words *das*, *dasma*, *dasra* have the sense of completing, fulfilling, accomplishing; *das* also has the related sense of getting exhausted. Yāska says that *śata* (hundred) derives from *daśadaśa* (ten tens), without further explanation. We mention here that due to their extreme antiquity, the etymology of Vedic words is often obscure; even ancient lexicographers had to use ingenious guesses rather than definitive knowledge.

The mystic civilisation of ancient Egypt also used base ten and had special pictorial symbols, with possible allegorical connotations, for powers of ten up to 10^7 (ten million). However, the Egyptian system of hieroglyphs did not anticipate the idea of "place value" or any analogue: to represent a number like "eight thousand", the symbol for thousand had to be repeated eight times. The Pythagoreans, inheritors of Egyptian and other mystic traditions, considered ten to be a perfect number (reminiscent of the completeness of *das*). They considered as sacred the *tectractys*, the triangular figure with ten special points arranged symmetrically in four rows containing one, two, three and four points ($1 + 2 + 3 + 4 = 10$) representing, among other things, the dimensions zero (point), one (line defined by two points), two (plane/triangle defined by three points) and three (tetrahedron defined by four points) respectively.

Exposition of Powers-of-Ten Principle in a Saṁhitā Treatise

A conscious enunciation of the decimal (powers-of-ten) principle can be seen in an incidental reference to it by the seer Medhātithi in the *Vājasaneyī Saṁhitā* (XVII.2) of the *Śukla Yajurveda*. This significant verse not only makes a list of terms for successive powers of ten up to 10^{12}, but also each of these terms is perceived, in fact defined, to be ten times the preceding term: *daśa daśa* a *śata* (hundred); *daśa śata* a *sahasra* (thousand); *daśa sahasra* an *ayuta* (ten thousand); *daśa ayuta* a *niyuta* (hundred thousand); *daśa niyuta* a *prayuta* (million); *daśa prayuta* an *arbuda* (ten million); *daśa arbuda* a *nyarbuda* (hundred million); *daśa nyarbuda* a *samudra* (thousand million); *daśa samudra* a *madhya* (ten thousand million); *daśa madhya* an *anta* (hundred thousand million); *daśa anta* a *parārdha* (English billion). These terms occur with some variations, and sometimes with further extensions, in other *Saṁhitā* and *Brāhmaṇa* texts. Terms for much higher powers of ten are mentioned in subsequent literature. An anonymous Jaina work *Amalasiddhi* has terms for all powers of 10 up to 10^{96} (*daśa-ananta*).

For expressing very large numbers in words, even the present English terminology (of using auxiliary bases like "thousand" and "million") is less satisfactory than the Sanskrit system of having a one-word term for each power of ten (up to some large power). This is effectively illustrated by G. Ifrah (2000, pp. 428–429) by comparing the verbal representations of the number 523622198443682439 in English, Sanskrit and other systems.

We mention here that for expressing high powers of ten, Jaina and Buddhist texts too use auxiliary bases like *sahassa* (thousand) and *koṭi* (ten million), resembling our English vocabulary; for instance, *prayuta* (10^6) would be *dasasatasahassa* (ten hundred thousand).

Use of Centesimal Enumeration

When convenient, centesimal (multiples of 100) scales have been used in India for expressing numbers larger than thousand. The *Ṛgveda* (I.53.9) describes 60099 as *ṣaṣṭiṁ sahasrā navatiṁ nava* (sixty thousands ninety nine); the *Taittirī ya Upaniṣad* (II.8) adopts a centesimal scale to describe different orders of bliss and mentions *Brahmānanda* (the bliss of Brahman) to be 100^{10} times a unit of human bliss; and in a dialogue in the Buddhist work *Lalitavistara* (c. 300 BCE), Lord Buddha lists numbers in multiples of 100 from *koṭi* (10^7) up to *tallakṣaṇa* (10^{53}). The *Rāmāyaṇa* uses a scale of hundred-thousand (10^5) to mention terms up to 10^{55} (*mahaugha*).

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Significance of Zero and Nine Numerals in Place Value Notation

In retrospect, a momentous step was taken by ancient Vedic seers (or their unknown predecessors) when they imparted single-word names to successive powers of ten, thus sowing the seeds of the decimal place value principle. The written decimal place value notation is simply a suppression of the place names (the terms for powers of ten) from the verbal decimal expression of a number, along with the replacement of word numerals by digits and the use of a zero symbol wherever needed. The transition to this notation needed, apart from symbols for the nine numerals, the idea and device of a zero symbol which will not carry any intrinsic positive value (unlike the nine primary numerals) but which will indicate the absence of any of the nine numerals in certain "places". In the expression "two thousand and eight", the word numeral "two" acquires the value "two thousand" by virtue of its being the adjective of "thousand", but in the notation 2008, which no longer carries the verbal support of a word like "thousand", the digit 2 can represent the value "two thousand" only because of the support of the symbol 0 which, by manning the hundred's and ten's places, enables 2 to get stationed in a position of "place value" thousand.

Zero has been denoted in India sometimes by a dot and sometimes by a small circle. The two most common Sanskrit words in Indian mathematics for zero are *śūnya* and *kha*. In *Ṛgveda*, the word *kha* refers to the hole in the nave of a wheel through which the axle runs. Its meanings include "cavity", "hollow", "aperture", "vacuity", "empty space", "sky" and the "great Void"; the *Bṛhādāraṇyaka Upaniṣad* identifies *kham* with Brahman, the Supreme Spirit. The word *śūnya* appears to have been derived from *śūna*, which means "hollowness", "emptiness" as well as "swelled", "increased", "grown"; there is another Vedic word *śuna* meaning "growth", "happiness", "auspiciousness" and "prosperity". Apart from the sense of emptiness, *kha* and *śūnya* have a subtle nuance of a potential for growth from a deficient state, a zero state, to a state of fullness. Their synonyms include not only words like *randhra* (small hole) but, more commonly, words like *ambara*, *ākāśa*, *abhra*, *gagana*, *antarikṣa* (all denoting the sky and the vastness of space), *pūrṇa* (whole, full, complete) and *ananta* (infinite). The *śūnya* or *kha* is not a mere void; it has immense possibilities and carries the essence of all that is yet to be formed or created. Using an allusion from Shakespeare, the American mathematician G. B. Halsted beautifully highlighted the analogy between the power of the place value of zero with this philosophical nuance (1912, p. 12):

> The importance of the creation of the zero mark can never be exaggerated. This giving to airy nothing, not merely a local habitation and a name, a picture, a symbol, but helpful power, is the characteristic of the Hindu race whence it sprang. It is like coining the Nirvana into dynamos. No single mathematical creation has been more potent for the general on-go of intelligence and power.

Apart from the two key ideas – the place value and the zero – the principle "one symbol for one digit" (reminiscent of the phonetic principle "one sound one symbol" of the Sanskrit alphabet) also contributes to the effectiveness of the decimal notation. A place value system was also used by Babylonians as early as 1700 BCE but with base 60 and with only two basic symbols (for ten and one respectively). It could not attain the economy and compactness of the decimal notation – fourteen symbols (five tens and nine ones) are needed to represent the Babylonian digit for 59.

An interesting feature of the Sanskrit numerals in the Devanāgarī script (now the most commonly used script for Sanskrit) is that each of them can be drawn in just one stroke (without lifting the pen/pencil from the paper), thereby imparting speed to the writing process.

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Appearance of Zero and Decimal Place Value Notation

While the structure of the Sanskrit vocabulary contains the key to the decimal system, in the absence of any direct palaeographic record, one does not know how numbers were written in the far-off Vedic times, when the idea of zero (and nine numerals) occurred in the context of number representation, and when the decimal place value notation was invented in India. The sacred Vedic literature was meant to be transmitted orally through a rigorous discipline in memorisation and not to be written down and copied – a step meant to eliminate errors in transmission – and composed accordingly. The purpose was served – we have practically the same versions of the classified Vedas all over India; however, we have to remain satisfied with information about their vocabulary and are in the dark about their script or notation.

It has been suggested on the basis of Indus valley metrology and apparent number symbols that a decimal system was in use in the Indus valley civilisation. In the epic *Mahābhārata* (III.134.16), there is a phrase *nava yogo gaṇanāmeti śaśvat*, "A combination of nine [digits] always [suffices] for any count [or calculation]". This incidental reference to the decimal place value notation occurs during the narration of a tale (III.132–134) involving ancient names like Uddālaka, Śvetaketu, Aṣṭāvakra, Janaka, et al., whose antiquity goes back to the *Brāhmaṇa* phase of the Vedic era. In fact, the word *śaśvat* (perpetual) has the nuance of "from immemorial time".

Some Sanskrit scholars see in the term *lopa* (elision, disappearance, absence) of Pāṇini's grammar treatise *Aṣṭādhyāyī* a concept analogous to zero as a marker for a non-occupied position and have wondered whether *lopa* led to the idea of zero in mathematics. Indeed, in a text *Jainendra Vyākaraṇa* of Pūjyapāda (c. 450 CE), the term "lopa" is replaced by *khaṃ*, a standard Sanskrit term for the mathematical zero. Pāṇini says *adarśanaṃ lopaḥ* (non-visibility is lopa) and uses it as a tool similar to the null operator in higher mathematics. Unfortunately, mathematical texts of the time of Pāṇini have not survived. (The date of Pāṇini is uncertain; most estimates vary from 700 BCE to 500 BCE.)

We now give an account of the scattered information we have regarding the (i) verbal mention of the chief ingredients of the decimal notation: zero symbol, places, nine numerals, etc., and (ii) actual occurrence of decimal notation in written documents.

Mention of Zero in Chandaḥ-sūtra of Pingalācārya

Two types of syllables *laghu* (short) and *guru* (long) form a Sanskrit metre; thus, any metre can be thought of as a sequence of binary numbers 0 and 1. In his prosody text *Chandaḥ-sūtra*, Piṅgalācārya gives rules to associate with each metre a unique number and to recover from a number the metre which it represents; they are essentially formulae for converting a binary representation of a number into its decimal representation and vice versa. The estimates of Piṅgalācārya's dates vary between 500 and 200 BCE, 300 BCE being the date used most frequently. Piṅgala's rules, enunciated two millenniums before Leibniz (1596 CE) introduced binary numbers, suggest a deep understanding of the principles underlying place value expansions.

Further, in two consecutive terse aphorisms – *rūpe śūnyam* and *dviḥ śūnye* – Piṅgala gives instructions involving the use of *dvi* (two) and *śūnya* (zero) as distinct labels. The choice of the labels suggests the prevalence of the mathematical zero and possibly a zero symbol. There is then a strong possibility that a decimal place value notation with zero was invented well before the time of Piṅgala; the quoted phrase from *Mahābhārata* supports this hypothesis.

For readers interested in the mathematical details, we mention here that the phrases *rūpe śūnyam* and *dviḥ śūnye* form part of Piṅgalācārya's algorithm to compute the total number of metres that can be formed using n syllables (i.e., the number 2^n) by a combination of squaring and doubling – the algorithm prescribes when to square and when to double, using the principles $2^m = \left(2^{\frac{m}{2}}\right)^2$ when m is even and

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$2^m = 2\left(2^{m-1}\right)$ when m is odd. The number n is to be halved continuously, and any stage where halving is done is to be marked with label two (*dvir ardhe*); when an odd number is reached, it is to be reduced by 1, and such a stage is to be labelled by zero (*rūpe śūnyam*). Doubling has to be done corresponding to each stage marked by zero (*dviḥ śūnye*) and squaring (multiplying by itself) at each of the other stages, i.e., where a number was halved (*tāvad ardhe tad guṇitam*). For instance, for $n = 11$, subtract 1 from 11 (put zero mark), then divide 10 by 2 (put two), then subtract 1 from 5 (mark it by zero), divide 4 by 2 (put two) and 2 by 2 (put two) and subtract 1 from 1 (mark it by zero). Reversing, we obtain successively 2, $2^{\mathbf{2}}$, $(2^{\mathbf{2}})^{\mathbf{2}}$, $(2^{\mathbf{2}})^{\mathbf{2}} \times 2$, $\left[(2^2)^2 \times 2\right]^2$ and $\left[(2^2)^2 \times 2\right]^2 \times 2\left(= 2^{11}\right)$. The reversal reminds one of a similar "descent"-like step in the later *kuṭṭaka* principle of Āryabhaṭa for solving linear indeterminate equations in integers (cf. Dutta, 2010).

Mention of Decimal Places and Zero in Jaina Texts

As pointed out by Datta (1929b), the concept of decimal place (*sthāna*) is explicitly mentioned in the Jaina text *Anuyogadvāra-sūtra* (c. 100 BCE). Here, the total number of human beings is described variously as "the product of 2^{64} and 2^{32}" (i.e., 2^{96}), as a number which "can be divided by two 96 times" and it is also said that this number, "when expressed in terms of denominations like *koṭi-koṭi* (10^{14})", "occupies 29 places (*sthāna*)". Indeed, the number $2^{96}(= 79228162514264337593543950336)$ requires precisely 29 digits in the decimal notation. Note that 0 occurs in the thousand's place in this 29-place decimal expansion. Thus, the reference to 29 places for this number indicates the prevalence, in some form, of the concept of zero as a placeholder.

Zero is explicitly mentioned in sixth-century Jaina texts. Jinabhadra Gaṇi (529–589 CE) abridges the verbal description of a number like 3200400000000 as "thirty-two, two zeros, four, eight zeros". The Jaina schools adopted the approximation formula $\sqrt{N} = a + \frac{r}{2a}$, where N is a positive integer, a the largest integer for which $a^2 < N$ and $r = N - a^2$. Jinabhadra Gaṇi describes the (approximate) square root $\sqrt{58545048750}$ (which is $241960\frac{407150}{483920}$ by the above formula) as "two hundred thousand forty one thousand nine hundred and sixty, *removing the zero*, the numerator is four-zero-seven-one-five and the denominator four-eight-three-nine-two". The reference to removal of zero (i.e., to cancel the common factor 10) suggests use of a zero symbol. Similar references occur in the work of Siddhasena Gaṇi.

Occurrence of Written Decimal Notation in a Manuscript

The ancient Brāhmī script (from which evolved the present scripts of various Indian languages) has symbols for the nine numerals, and also separate symbols for multiples of 10 (up to 90) and multiples of 100 (up to 900). There are inscriptions from the third century BCE in the Brāhmī script (including the rock edicts of Emperor Aśoka) where numbers are inscribed in symbols using base 10 but not in place value notation. The modern nine numerals are derived from the Brāhmī numerals.

We directly see numbers written in decimal place value notation in the most ancient Indian mathematics manuscript discovered so far – the *Bakṣālī Manuscript* in the Gāthā dialect (combination of Sanskrit and Prakrit) in Śāradā script. It contains mathematical advancements of the period prior to Āryabhaṭa (fifth century CE) but not the topics (like solutions of linear indeterminate equations) seen first in the treatises of Āryabhaṭa and his successors. The discovered manuscript (dated from around the eighth century CE) is a copy of an earlier document estimated to have been composed between the second century BCE and the seventh century CE (200 BCE – 200 CE by L. Gurjar, 200 CE –300 CE by B. Datta, 200 CE– 400 CE by R. Hoernle and many others, seventh century CE by T. Hayashi).

The nine numerals we see in the manuscript appear to have evolved from the Brāhmī numerals and closely resemble our present Devanāgarī numerals; zero is denoted by a small circle. The numbers

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occurring in this manuscript (in decimal numerals) include the six-digit number 846720, apart from three-digit numbers 330 and 947 (cf. Joseph, 2000, p. 241).

Exposition of Decimal Place Value in a Mathematics Treatise of Āryabhaṭa

The idea of *sthāna* as a denominational place is taught in the *Gaṇita* (mathematics) chapter of the fifth century treatise *Āryabhaṭīya* (499 CE) of Āryabhaṭa. After an invocation in the first verse, the chapter narrates (in Verse two) the first ten notational places: *eka*, *daśa*, *śata* (hundred), *sahasra* (thousand), *ayuta* (ten thousand), *niyuta* (hundred thousand), *prayuta* (million), *koṭi* (ten million), *arbuda* (hundred million) and *vṛnda* (thousand million) and declares them as being *sthānāt sthānaṁ daśaguṇaṁ syāt*, "from *place* to *place*, it is ten times [the preceding]". Subsequent authors list more places (usually 18, sometimes 24). In an earlier millennium, Vedic authors like Medhātithi had already enunciated the "powers-of-ten" principle "each ten times the preceding". In the above verse of Āryabhaṭa, we see a clear formulation of the next crucial step for decimal place value notation – visualising each power of ten as a *sthāna* (place). The verse is only an exposition and not an announcement of a new concept. In the preceding verse (Verse one of *Gaṇita*), Āryabhaṭa acknowledges that he is recording the "knowledge revered at Kusumapura" (Pāṭaliputra, near modern Patna).

Āryabhaṭa's methods in *Gaṇita* for extraction of the square root and the cube root of a number, which are slight variants of the modern methods, make intricate uses of the concepts of place value and zero. The fact that these methods are briefly stated without elaboration indicates that the decimal place value notation with zero had already become firmly entrenched in Indian arithmetic. His alphabetical coding of large numbers (Verse two of the *Gītikā* chapter of *Āryabhaṭīya*), using a centesimal system in which the consonants of the Sanskrit alphabet act as digits and the vowels play the role of place value (Dutta, 2006), reiterate how strong was his mastery over all the ideas encapsulated in the decimal notation.

Exposition of Zero as an Integer in a Mathematics Treatise of Brahmagupta

For clarity in our understanding of the development of zero in mathematics, a distinction is made between two of its important features: (i) the notational innovation of the zero symbol as a placeholder in a place value numeral system; and (ii) the abstract conception of the integer zero in a number system amenable to arithmetic operations. Aspect (ii) is central in modern mathematics with its emphasis on algebraic structures. (Note that the two aspects are mathematically equivalent, for the decimal notation $a_n \ldots a_1 a_0$ corresponds to the polynomial expression $10^n a_n + \ldots + 10a_1 + a_0$ whose coefficients $a_n \ldots, a_1, a_0$ are regarded as integers between 0 and 9.)

The profound step (ii) of elevating zero to an algebraic number (*saṅkhyā*) (and not a mere placeholder) was taken in India. In a section of Chapter XVIII of his treatise *Brāhma Sphuṭa Siddhānta* (628 CE), Brahmagupta defines zero as $a - a$ and gives an elaborate treatment of the four fundamental operations involving numbers – positive, negative and zero – treating all on the same footing. His rules amount to the sophisticated idea of giving integers a ring structure with zero as the additive identity. This comprehensive treatment of zero and negative numbers was an important pillar for the foundations of algebra established by Brahmagupta in the above-mentioned chapter.

The use or mention of zero as a number in arithmetic can be glimpsed earlier in the *Bakṣālī Manuscript*, in the sixth-century treatise *Pañcasiddhāntikā* of Varāhamihira and in the early seventh-century commentary on *Āryabhaṭīya* by Bhāskara I.

That step (ii) needs a conceptual leap from (i) can be seen from the fact that although the decimal notation requires ten symbols, ancient and medieval scholars often refer to it as a system with "nine figures" indicating that the placeholder zero symbol did not always receive the dignity accorded to the other nine numerals. We mention here that a partial use of the placeholder principle (i) had also emerged in some form among late Babylonians during 700–400 BCE; the Mayan culture (fourth century CE) too used

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zero as a placeholder and conceived of zero as an ordinal preceding one – the first day of their Haab calendar was day “zero”.

Occurrence of Written Decimal Notation in Inscriptions

A copper plate from Gujarat of 595 CE is so far the oldest discovered inscription depicting a number (346) in decimal place value notation. For numbers with a zero digit, the earliest known examples are certain Sanskrit inscriptions dated 683–687 CE from places in modern Cambodia and Indonesia (then a part of Greater India) which give their Śaka dates 605, 606 and 608 in decimal notation.

Among early inscriptions within present India which represent numbers in decimal notation with a zero digit, the most famous is the Gwalior inscription of King Bhojadeva, dated 870 CE, which depicts the numbers 50 and 270. But older inscriptions with zero in decimal notation have also been reported (cf. Datta and Singh, 1935; Singh, 2007), e.g., the number 30 in Trilingi plates (690 CE) and Ragholi plates (eighth century) and the number 201 in the Khandela inscription (807 CE). It is mentioned in (Mukherjee, 1977) that an inscription dated 672 CE of Āditya Sena at Shahpur (Bihar) depicts the number 60 with zero represented by a small dot. A list of inscriptions from 578 CE which depict the zero symbol, not necessarily as part of decimal notation, is presented in (Singh 2007).

Alphabetical Decimal Notation

A decimal place value notation called *kaṭapayādi* made use of letters of the Sanskrit alphabet in place of numerical figures. The letters were usually written in ascending powers of 10 (the digit in the unit’s place written first followed by the digit in the ten’s place to its right, and so on). The system was used in Kerala to compose *vākyas* (sentences) recording planetary positions at regular intervals. The earliest known treatise of this type is the *Candra-vākyāni* of Vararuci who is regarded in the astronomy tradition of Kerala as belonging to the fourth century CE. The *kaṭapayādi* system can be seen in several subsequent treatises of south India, like the *Grahacāranibandha* of Haridatta (c. 600 CE).

The *kaṭapayādi* provided scope for skilled authors to compose brief but pleasant-sounding chronograms, often with connected meanings; sometimes, the Kaṭapayādi value of a name would encode one of its numerical features.

For readers familiar with the Sanskrit alphabet, we describe one of the four versions of the *kaṭapayādi* system. The nine consonants from *k* to *jh*, as also the nine consonants from *ṭ* to *dh*, denoted (in usual order) the digits from 1 to 9; the five consonants from *p* to *m* denoted the digits 1 to 5, while the eight unclassed consonants from *y* to *h* denoted 1 to 8. Thus, 1 could be denoted by any of the letters *ka, ṭa, pa, ya* (hence the name of the scheme). The consonants *ñ* and *n* and all pure vowels not preceded by a consonant (i.e. occurring at the beginning) denoted 0. A vowel joined to a consonant, or a consonant not joined to a vowel, did not carry numerical value. In a conjoined consonant, only the last one would denote a digit. For instance, the *kaṭapayādi* word *tatvāloke* denotes the number 1346 (*ta* = 6, *tvā* = 4, *lo* = 3, *ke* = 1).

Decimal Place Value through Word Numerals

As most ancient Indian scientific treatises were composed in verses, there was a tradition of expressing numbers verbally by the decimal place value principle including zero using word numerals instead of symbols. The *Bakṣālī Manuscript* has both word numerals and representations in symbols, the word numerals in this treatise being simply the usual number names. Another early instance cited by Ifrah (2000) is a Jaina cosmology text *Lokavibhāga* dated 458 CE.

In a popular system of word numerals, which emerged by the third century CE, the word name for a digit (or even a number) *n* was chosen to be a well-known object or idea which usually occurs with frequency *n* (or has *n* components). Such a word numeral was called *bhūtasaṅkhyā*, i.e., a number (*saṅkhyā*) using an object (*bhūta*) illustrating the number. For instance, *candra* (Moon) or *mahī* (Earth)

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stood for one, *netra* (eyes) for two, *guṇa* (sattva-rajas-tamas) for three, *yuga* (satya-tretā-dvāpara-kali) for four, *indriya* (sense-organs) for five, *rasa* (taste) for six and so on; synonyms for the sky were used for zero. (Here, *rasa* refers to the six food tastes; in Sanskrit aesthetics, *rasa* refers to various dramatic moods.)

The place values of word numerals in a *bhūtasahkhyā* increased from left to right. For instance, Bhāskarācārya describes his year of birth 1036 Śaka Era (1114 CE) as *rasa-guṇa-pūrṇa-mahī*.

This system can be seen in the *Yavanajātaka* of Sphujidhvaja (270 CE), in the *Sūryasiddhānta* and *Puliśasiddhānta* (original versions composed by the fourth century) and in later astronomy and mathematical treatises, and in the *Agni-Purāṇa* (portions of which were composed within the fourth century). Word numerals are also found in certain Sanskrit inscriptions of the seventh century in the Far East.

B. Datta feels that decimal place value notation had been in common use long before the idea of applying the place value principle to a system of word names was conceived. In the beginning, number names were used as word numerals. *Bhūtasaṅkhyā* replaced number names as the numerous choices for any digit helped in versification and in infusing a poetic charm to the technical presentations. Bhāskarācārya's phrase *rasa* (taste; delight) – *guṇa* (quality) – *pūrṇa* (full) – *mahī* (Earth) for the number 1036 has a sweet meaning: "The Earth is full of delight".

Place Value and Zero as Metaphors in Philosophy and Literature

By perhaps the first century CE, the place value notation had become so well established among the cultured elite in India that the idea could be used as a metaphor in works on philosophy. A reference to place value occurs in a work of the Buddhist philosopher Vasumitra (probably first century CE):

> When [the same] clay counting-piece is in the place of units, it is denoted as one, when in hundreds, one hundred. (Quoted in (Plofker, 2009, p. 46))

Both the *Vyāsa-bhāṣya* (probably fifth century CE) on the *Yoga-sūtra* of Patañjali and *Sārīraka-bhāṣya* of Śaṅkarācārya (c. 750 CE) use the following simile (translation follows the latter):

> Just as the same stroke (or figure) conveys different ideas such as a unit, a ten, a hundred, a thousand, etc., according to the place in which it is set down, ... (Original verse with translation quoted in Ganguli (1932, p. 255))

The romantic literary work *Vāsavadattā* of Subandhu (sixth century) uses the simile of a zero-symbol *śūnya-bindu* to describe the stars.

The zero of the decimal system has also been used to explain spiritual principles. Indian philosophy regards the universe as a manifestation of The One Reality, the *Pūrṇa*, but emphasises that without the realisation of (or at least a conscious link with) the One Brahman, the transient universe is an unreal vacant nought, a *śūnya*; just as many zeros form a large number if preceded by 1; but without that 1, a large collection of zeros amounts to nothing. In the utterance of Sri Ramakrishna (1884), translated by Swami Nikhilananda (1944, p. 643):

> First realize God, then think of the creation and other things. . . . If you put fifty zeros after one, you have a large sum; but erase the one and nothing remains. It is the one that makes the many. First one, then many.

Reverence for the Invention

Indians regarded the invention of the decimal system as something special; it had a status similar to that of an inspired revelation. The Arab historian Abul Hasan Al-Masūdī, who visited India during the tenth century, writes (943 CE), "A congress of sages at the command of the Creator Brahmā invented the nine figures . . ." (Datta & Singh, 1935, p. 97). In his popular text *Līlāvatī*, the mathematician-astronomer Bhāskarācārya (1150 CE) makes a special acknowledgement that the decimal system of place values was conceived by the ancients to simplify practical computations.

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Impact of Decimal System on Ancient Indian Mathematics

The decimal system is largely responsible for the excellence attained by Indian mathematicians in the fields of arithmetic, algebra and astronomy. Thanks to the notation, Indians could develop efficient methods for the basic arithmetic operations (which were slight variants of our present methods) at an early stage.

The decimal system (both oral and written) enabled Indians to express large numbers effortlessly, right from the Vedic Age. This traditional facility with large numbers enabled Indians to work with large time frames in astronomy which helped them obtain accurate results. Indian algebraists could venture into problems on equations whose solutions involve large integers. Bhāskarācārya (1150 CE) discussed methods for finding integer solutions of an equation like $61x^2 + 1 = y^2$; the smallest pair of integers x, y satisfying this equation turns out to be $x = 226153980$, $y = 1766319049$. Five centuries later, soon after the decimal system had been standardised in Europe, this equation would again be highlighted by Fermat (1657), heralding the advent of modern number theory.

The decimal system has a dormant algebraic character. The decimal expansion of integers must have influenced the development of the near-analogous algebra of polynomials by Brahmagupta (628 CE) and the power series expansions of trigonometric functions by Mādhava (fourteenth century CE). A millennium after Brahmagupta, Isaac Newton (a pioneer in the study of polynomials and power series in modern Europe) emphasised (1671) that the arithmetic of decimal numbers provides a fruitful model for developing the arithmetic operations on algebraic expressions in variables. Newton had observed (quoted in (Divakaran, 2010, p. 296)):

> "Since the operations of computing in numbers and with variables are closely similar – indeed there appears to be no difference between them except in the characters by which quantities are denoted, definitely in the one case, indefinitely so in the latter – I am amazed that it has occurred to no one . . . to fit the doctrine recently established for decimal numbers in similar fashion to variables, especially since the way is then open to more striking consequences. For since this doctrine in species has the same relationship to Algebra that the doctrine in decimal numbers has to Arithmetic, its operations of Addition, Subtraction, Multiplication, Division and Root-extraction may easily be learnt from the latter's . . ."

The polynomial, power series and p-adic expansions are conscious generalisations of the decimal expansion.

Recursive principles dominate Indian mathematical thought and are prominent features of some of its greatest achievements like the solutions of indeterminate equations and the work of the Kerala school (cf. Dutta, 2010; Divakaran, 2010). The facility with recursive methods is another outcome of the decimal representation of numbers, the first known example of recursive construction.

Note that the Vedic verbal system itself has this polynomial and recursive aspects with their striking mathematical consequences.

Transmission of Decimal Notation

In the absence of records of commercial transactions during the Dark Ages of Europe and the paucity of extant scientific treatises of the period, one cannot ascertain when the Indian notation first reached the West. The Indian numerals have been found in tenth-century manuscripts of the Geometry of Boethius (c. 500 CE), the Latin philosopher. It is believed by many historians that sometime in the fifth century CE, the Indian numerals reached Alexandria – possibly through traders and merchants – and spread farther westward. That the fame of the Indian decimal notation had reached the banks of the Euphrates by the

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early seventh century can be seen in a passing reference by the Syrian astronomer–monk Severus Sebokht (662 CE):

> I shall not now speak of the knowledge of the Hindus, . . . of their subtle discoveries in the science of astronomy – discoveries even more ingenious than those of the Greeks and Babylonians – of their rational system of mathematics, or of their methods of calculation which no words can praise strongly enough – I mean the system using nine symbols. (Quoted in (Basham, 1954, p. vi))

A manuscript *Codex Vigilanus* of 976 CE (now kept in a museum of Madrid) by Vigila, a monk in a Spanish monastery, also records a similar appreciation:

> The Indians have an extremely subtle intelligence, and when it comes to arithmetic, geometry and other such advanced disciplines, other ideas must make way for theirs. The best proof of this is the nine figures with which they represent each number no matter how large. (Quoted in (Ifrah, 2000, p. 362; Joseph, 2000, p. 313))

Numerous Arabic texts were written on the decimal notation and Indian arithmetic, beginning with a treatise composed by Al-Khwārizmī around 820 CE. The original text of Al-Khwārizmī is lost, but its Latin translation *Liber Algorismi de numero Indorum* (The Book of Al-Khwārizmī on Indian numerals), composed in the early twelfth century, still survives. The following lines, quoted in (Bag & Sharma, 2003, p. 168), convey some flavour of Al-Khwārizmī's exposition on the place value principle with nine numerals and zero:

> When I saw that Indians composed out of IX letters any number due to the position established by them, I desired to discover, God willing, what becomes of those letters to make it easier for the student. . . . Thus they created . . . IX letters, . . . The beginning of the order is to the right side of the writer, and this will be the first of them consisting of unities. If instead of unity they wrote X, it stood in the second digit, and their figure was that of unity, they needed a figure of tens similar to the figure of unity so that it becomes known that this was X. Thus they put before it one digit and wrote in it a small circle "o", so that it would indicate that the place of unity is vacant.

The Latin translation made a significant impact in the West and triggered several adaptations. Al-Khwārizmī's name got so closely associated in Europe with the new arithmetic based on the Indian decimal numerals that the Latin form of his name *algorismus* was given to any treatise on computational mathematics – especially those that made use of the zero. (This is the genesis of the word "algorism" which later became "algorithm".) The numerals were explicitly called *hindisah* (Indian) by Al-Nadīm (c. 987 CE). The Arabic translation *sifr* (meaning "empty") of the Sanskrit word *śūnya* became *cifra* in Spanish (source of the word *cipher*) and *chiffre* in French, *zephirum* in Medieval Latin, *ziffer* in German and *zero* in English.

A vigorous effort to bring the Indian decimal system into common use in the West was made by Leonardo Fibonacci of Pisa (1180–1240). Acquainted with various systems of calculations, Fibonacci found all computational systems deficient compared to the Indian system: "*quasi errorem computavi respectu modi indorum*" (Datta & Singh, 1935, p. 95). His influential work *Liber Abaci* (Book of Computations) in 1202 (rewritten in 1228), based entirely on Indian numerals, prepared the ground for the spread and adoption of the Indian decimal system and computation methods. Several subsequent thinkers in Europe too advocated the system. A woodblock engraving *Margarita Philosophica* of Gregor Reisch (1503) contrasts the old and new systems by depicting a gloomy Pythagoras toiling with a counting board, a cheerful and serene Boethius comfortable with his computations (using the decimal system) and Lady Arithmetic giving her smiling approval of Boethius, the Indian numerals adorning her dress (see Fig. 1). In another illustration by the Nuremberg artist Hans Sebald Beham (1550), Winged Arithmetic is shown turning her back on the counting board and pointing emphatically to a tablet with the new Indian numerals.

However, Roman numerals had got so deep rooted in Europe that it took around five centuries for the decimal notation to gain universal acceptance. Initially, the new system "was too advanced for the

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Fig. 1 Wood-block engraving from Gregorius Reisch, *Margarita Philosophica* (Freiburg, 1503). Lady Arithmetic (standing in the centre) gives her judgment by smiling on the arithmetician (to our left, her right) working with Arabic numerals and the zero (the numerals also adorn her dress). The quarrel of the abacists and algorists is over, and the latter have won

merchants and too novel for the universities" (Datta & Singh, 1935, p. 95), and there was a determined resistance against it from the abacists. With passage of time, more people began to realise its importance, and the decimal notation eventually came into common use in the seventeenth century.

The introduction of the decimal notation was one of the key factors that triggered the commercial, mathematical and scientific renaissance in Europe. It immensely simplified arithmetic; all modern methods of fundamental arithmetic operations are dependent on the principles of decimal notation. The Australian historian A. L. Basham observed about this priceless gift of India (Basham, 1954, pp. 495–496):

> Most of the great discoveries and inventions of which Europe is so proud would have been impossible without a developed system of mathematics, and this in turn would have been impossible if Europe had been shackled by the unwieldy system of Roman numerals. The unknown man who devised the new system was from the world's point of view, after the Buddha, the most important son of India. His achievement, though easily taken for granted, was the work of an analytic mind of the first order, and he deserves much more honour than he has so far received.

Epilogue

The history of the decimal system may be summarised in the words of Swami Vivekananda (from a speech in 1895 titled "India's Gift to the World"):

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> . . . the ten numerals, the very cornerstone of all present civilization, were discovered in India, and are, in reality, Sanskrit words. (Swami Vivekananda, 2012, p. 511)

References

Bag, A. K., & Sarma, S. R. (Eds.). (2003). *The Concept of Śūnya*. New Delhi, India: Indira Gandhi National Centre for the Arts, Indian National Science Academy and Aryan Books International.

Basham, A. L. (1954). *The wonder that was India*. London: Sidgwick and Jackson.

Bavare, B., & Divakaran, P. P. (2013). Genesis and early evolution of decimal enumeration: Evidence from number names in Ṛgveda. *Indian Journal of History of Science, 48*(4), 535–581.

Coomaraswamy, A. (1934). Kha and Other Words denoting "Zero" in connection with the metaphysics of space. *Bulletin of the School of Oriental Studies, University of London, 7*(3), 487–497. (Reprinted in *Perceptions of the Veda*, 2000, New Delhi: IGNCA Publications).

Dantzig, T. (1930). *Number: The Language of Science*. New York: Macmillan.

Datta, B. (1926[1931]). Early literary evidence of the use of the zero in India. *American Mathematical Monthly*, *33*, 449–454; *38*, 566–572.

Datta, B. (1929a). The Bakhshālī mathematics. *Bulletin of the Calcutta Mathematical Society, 21*, 1–60.

Datta, B. (1929b). The Jaina school of mathematics. *Bulletin of the Calcutta Mathematical Society, 21*, 115–145.

Datta, B. (1986). Vedic mathematics. In P. Ray & S. N. Sen (Eds.), *The Cultural Heritage of India* (Vol. VI). Calcutta, India: The Ramakrishna Mission Institute of Culture.

Datta, B., & Singh, A. N. (1935). *History of Hindu Mathematics* Part I. India: Motilal Banarsidass. (Reprinted by Asia Publishing House (1962) & Bharatiya Kala Prakashan (2004).)

Divakaran, P. P. (2010). Notes on Yuktibhāṣā: Recursive methods in Indian mathematics. In C. S. Seshadri (Ed.), *Studies in the History of Indian Mathematics*. New Delhi, India: Hindustan Book Agency.

Divakaran, P. P. Excerpts from a forthcoming book.

Dutta, A. K. (2002). Turning points in mathematics – Historical perspectives II. *Srinvantu, XLVXIV*(2), 49–64.

Dutta, A. K. (2006). Alphabetical coding of large numbers. *Resonance, 11*(4), 69–74.

Dutta, A. K. (2008). Powers of ten: Genesis and mystic significance. *Srinvantu, 58*(2), 44–52. (Reprinted in *Mother India* (2009), LXII(11), 957–967; *Vedic Venues* (2013), *2*, 43–56).

Dutta, A. K. (2010). Kuṭṭaka, Bhāvanā and Cakravāla. In C. S. Seshadri (Ed.), *Studies in the History of Indian Mathematics*. New Delhi, India: Hindustan Book Agency.

Ganguli, S. (1932[1933]). The Indian origin of the modern place-value arithmetical notation. *American Mathematical Monthly*, *39*(5), 251–256; *39*(7), 389–393; *40*(1), 25–31; *40*(3), 154–157.

Gupta, R. C. (1983a). Decimal denominational terms in ancient and medieval India. *Gaṇita Bhāratī, 5*, 8–15.

Gupta, R. C. (1983b). Spread and triumph of Indian numerals. *Indian Journal of History of Science, 18*(1), 23–38.

Gupta, R. C. (2001). World's longest lists of decuple terms. *Gaṇita Bhāratī, 23*, 83–90.

Halsted, G. B. (1912). *On the foundation and technique of Arithmetic*. Chicago: The Open Court Publishing Company.

Ifrah, G. (2000). *The Universal History of Numbers*. New York: John Wiley and Sons.

Joseph, G. G. (2000). *The Crest of the Peacock*. London: Penguin.

Joseph, G. G. (2002). The enormity of zero. *Mathematics Teacher, 38*, 108–125.

Monier-Williams, M. (1899). *Sanskṛit-English Dictionary.* Oxford, UK: Clarendon.

DOI 10.1007/978-94-007-3934-5_10321-1

Mukherjee, R. N. (1977). Background to the discovery of the symbol for zero. *Indian Journal of History of Science, 12*(2), 226–231.

Mukhopadhyay, P. (2009). Concept of Śūnya in Indian antiquity. *Journal of Pure Mathematics, 26*, 15–49.

Pandit, M. D. (1990). *Zero in Pāṇini*. Pune, India: University of Poona.

Pandit, M. D. (1993). *Mathematics as known to the Vedic Saṁhitās*. Delhi, India: Sri Satguru Publications (Indian Books Centre).

Patwardhan, K. S., Naimpally, S. A., & Singh, S. L. (Eds.). (2001). *Līlāvatī of Bhāskarācārya*. Delhi, India: Motilal Banarsidass.

Plofker, K. (2009). *Mathematics in India*. Princeton, New Jersey: Princeton University Press. (Reprinted by Hindustan Book Agency (2012)).

Seidenberg, A. (1962). The ritual origin of counting. *Archive for History of Exact Sciences, 2*(1), 1–40.

Sen, S. N. (1971). Mathematics. In D. M. Bose, S. N. Sen, & B. V. Subbarayappa (Eds.), *A Concise History of Science in India*. New Delhi, India: Indian National Science Academy.

Sen, S. N., & Bag, A. K. (1986). Post-vedic mathematics. In P. Ray & S. N. Sen (Eds.), *The Cultural Heritage of India* (Vol. VI). Calcutta, India: Ramakrishna Mission Institute of Culture.

Sharma, R. S. (Ed.). (1966). *Brāhma Sphuṭa Siddhānta of Brahmagupta*. New Delhi, India: Indian Institute of Astronomical and Sanskrit Research.

Shukla, K. S., & Sarma, K. V. (Eds.). (1976). *Āryabhaṭīya of Āryabhaṭa*. New Delhi, India: Indian National Science Academy.

Singh, A. K. (2007). Zero in early Indian inscriptions. *Purātattva, 38*, 121–126.

Sri Aurobindo. (1972). *The Secret of the Veda*. Sri Aurobindo Birth Centenary Library (SABCL) Vol. 10. Pondicherry, India: Sri Aurobindo Ashram. (Originally appeared in the *Arya* in 24 instalments from August 1914 to July 1916.)

Swami Nikhilananda (1944). *The Gospel of Sri Ramakrishna* (4th ed.) (1964). Madras, India: Sri Ramakrishna Math. Slightly edited translation of *Sri Ramakrishna Kathamrita* (in Bengali) by M. Gupta.

Swami Vivekananda (2012). *Complete works* (10th ed., Vol. II). Kolkata, India: Advaita Ashram.